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THIRD-ORDER RELATIVISTIC DYNAMICS: CLASSICAL SPINNING PARTICLE TRAVELLING IN A PLANE

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Mathisson's 'new mechanics' of a relativistic spinning particle is shown to follow, in the case of planar motion, from only general requirements of relativistic invariance and of the dependence on third order derivatives along with the 'variationality' feature. The hamiltonian counterpart ultimately recovers the Dixon equations for this case with the Pirani supplementary condition.

Key words: *Lagrangian, Hamiltonian, Ostroghrads'kyj_mechanics, Classical_spin, Relativistic_top.*

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1. INTRODUCTION

Following the updating tendencies in the formal theory of variational calculus promoted by the development of the intrinsic differential geometry as well as global analysis, many authors have revisited the Ostroghrads'kyj mechanics with higher derivatives. The subject develops continuously and, surprisingly enough, models of physical meaning breed (see [1] and references in [2]).

In this paper we consider a model third-order dynamics of a classical particle which, although restricted to the unrealistic three-dimensional flat space-time, provides an instructive example of how new hamiltonian systems of physical meaningfulness may arise from higher-order variational calculus. This example admits a comprehensive solution of the involved variational inverse problem for invariant third-order equation of motion. It turns out that the equation thus obtained may be interpreted as yet another description of a planar motion of the classical spinning particle in special relativity. Taking into account that the general-relativistic equation of motion of the gravitational dipole particle admits, among others, also a solution of only two degrees of freedom (see [3,4]), we hope that the results of present

investigation may contribute to the future lagrangian and hamiltonian formulation of the general Mathisson equation [5] in the realistic four-dimensional curved space-time:

$$m_0 \frac{D\mathfrak{u}_\mathfrak{p}}{d\tau} - \mathfrak{S}_{\mathfrak{pq}} \frac{D^2\mathfrak{u}^\mathfrak{q}}{d\tau^2} = \frac{1}{2} \mathfrak{u}^\mathfrak{m} \mathfrak{R}_{\mathfrak{mpnq}} \mathfrak{S}^{\mathfrak{nq}} \,, \qquad (1)$$
$$\mathfrak{u}_\mathfrak{q} \mathfrak{u}^\mathfrak{q} = 1 \,.$$

If only a *free* and *planar* motion is going to be considered, the equation (1) splits into the following two ($\mu, \nu = 0, 1, 2$):

$$m_0 \dot{u}_\mu - S_{\mu\nu} \ddot{u}^\nu = 0 \,, \qquad (2)$$
$$\mathfrak{S}_{3\mu} \ddot{u}^\mu = 0 \,. \qquad (3)$$

We shall demonstrate later that the equation (2) may be cast into the form of being the *only* third-order relativistic equation admitting a lagrangian description.

Applying to it a kind of the 'hamiltonization' prescription of [6] then yields an equivalent to the Dixon equations [8] adopted to the case presently considered here, from which the equation (2) follows in turn, provided the Pirani supplementary condition

$$\mathfrak{u}_\mathfrak{q} \mathfrak{S}^{\mathfrak{pq}} = 0 \qquad (4)$$

is in force.

2. LAGRANGIAN DESCRIPTION

Necessary and sufficient conditions to the existence of a Lagrange function for a third-order differential equation

$$\mathsf{A} \,.\, \mathsf{v}'' + (\mathsf{v}' . \,\partial_\mathsf{v})\, \mathsf{A} \,.\, \mathsf{v}' + \mathsf{B} \,.\, \mathsf{v}' + \mathsf{c} \;=\; \mathbf{0} \qquad (5)$$

were established in [9]. They are expressed by means of the following system of partial differential equations in the independent variables t, x^a, and v^a

$$\begin{aligned}
&(i) \quad \partial_{\mathsf{v}^{[a}} \mathsf{A}_{bc]} = 0 \\
&(ii) \quad 2\, \mathsf{B}_{[ab]} - 3\, \mathbf{D_1} \mathsf{A}_{ab} = 0 \\
&(iii) \quad 2\, \partial_{\mathsf{v}^{[a}} \mathsf{B}_{b]c} - 4\, \partial_{\mathsf{x}^{[a}} \mathsf{A}_{b]c} + \partial_{\mathsf{x}^c} \mathsf{A}_{ab} + 2\, \mathbf{D_1} \partial_{\mathsf{v}^c} \mathsf{A}_{ab} = 0 \\
&(iv) \quad \partial_{\mathsf{v}^{(a}} \mathsf{c}_{b)} - \mathbf{D_1} \mathsf{B}_{(ab)} = 0 \\
&(v) \quad 2\, \partial_{\mathsf{v}^c} \partial_{\mathsf{v}^{[a}} \mathsf{c}_{b]} - 4\, \partial_{\mathsf{x}^{[a}} \mathsf{B}_{b]c} + \mathbf{D_1}^2 \partial_{\mathsf{v}^c} \mathsf{A}_{ab} + 6\, \mathbf{D_1} \partial_{\mathsf{x}^{[a}} \mathsf{A}_{bc]} = 0 \\
&(vi) \quad 4\, \partial_{\mathsf{x}^{[a}} \mathsf{c}_{b]} - 2\, \mathbf{D_1} \partial_{\mathsf{v}^{[a}} \mathsf{c}_{b]} - \mathbf{D_1}^3 \mathsf{A}_{ab} = 0 \,.
\end{aligned} \qquad (6)$$

We recall that the skew-symmetric matrix A, the matrix B, and the column vector c depend on the variables t, x, $\mathsf{v} = d\mathsf{x}/dt$. The differential operator $\mathbf{D}_{\mathbf{1}}$ denotes the first order generator of the Cartan distribution,

$$\mathbf{D}_{\mathbf{1}} = \partial_t + \mathsf{v} \,.\, \boldsymbol{\partial}_{\mathsf{x}} \,.$$

Any third-order Euler-Poisson equation (i.e. that of some variational origin) in two space dimensions fits into the form (5).

We are interested only in the equations bearing the Poincaré symmetry with the infinitesimal generator X parametrized by means of a skew-symmetric matrix $\boldsymbol{\Omega}$ and some vector $\boldsymbol{\pi}$:

$$\begin{aligned} X = & -(\boldsymbol{\pi} \cdot \mathsf{x})\, \partial_t + g_{00}\, t\, \boldsymbol{\pi} \,.\, \boldsymbol{\partial}_{\mathsf{x}} + \boldsymbol{\Omega} \cdot (\mathsf{x} \wedge \boldsymbol{\partial}_{\mathsf{x}}) \\ & + g_{00}\, \boldsymbol{\pi} \,.\, \boldsymbol{\partial}_{\mathsf{v}} + (\boldsymbol{\pi} \cdot \mathsf{v})\, \mathsf{v} \,.\, \boldsymbol{\partial}_{\mathsf{v}} + \boldsymbol{\Omega} \cdot (\mathsf{v} \wedge \boldsymbol{\partial}_{\mathsf{v}}) \\ & + 2\, (\boldsymbol{\pi} \cdot \mathsf{v})\, \mathsf{v}' \,.\, \boldsymbol{\partial}_{\mathsf{v}'} + (\boldsymbol{\pi} \cdot \mathsf{v}')\, \mathsf{v} \,.\, \boldsymbol{\partial}_{\mathsf{v}'} + \boldsymbol{\Omega} \cdot (\mathsf{v}' \wedge \boldsymbol{\partial}_{\mathsf{v}'}) \,. \end{aligned}$$

The centered dot symbol denotes the inner product of vectors or tensors; the lowered dot symbol denotes the contraction of a row-vector and the subsequent column-vector, or (sometimes) the contraction of a matrix and the subsequent column-vector. In order to chose a convenient expression of the symmetry concept we introduce a vector differential form $\boldsymbol{\epsilon}$, associated with the equation (5):

$$\begin{aligned} \epsilon_{\boldsymbol{a}} &= \mathsf{A}_{\boldsymbol{ab}} \mathrm{d}\mathsf{v}'^{\boldsymbol{b}} + \mathsf{k}_{\boldsymbol{a}} \mathrm{d}t \,, \\ & \qquad \mathsf{k} = (\mathsf{v}' \cdot \boldsymbol{\partial}_{\mathsf{v}})\, \mathsf{A}\mathsf{v}' + \mathsf{B}\mathsf{v}' + \mathsf{c} \,. \end{aligned} \tag{7}$$

Now it is possible to cast the idea of the symmetry of the equation (5) into the framework of the concept of exterior differential system invariance. The system in case is generated by the vector valued Phaff form $\boldsymbol{\epsilon}$ and the contact vector valued differential forms

$$\mathbf{d}\mathsf{x} - \mathsf{v}\mathbf{d}t \,, \qquad \mathbf{d}\mathsf{v} - \mathsf{v}'\mathbf{d}t \,. \tag{8}$$

Let $X(\boldsymbol{\epsilon})$ denote the Lie derivative of the vector valued differential form $\boldsymbol{\epsilon}$ along the vector field X. The invariance condition consists in that there may be found some matrices $\boldsymbol{\Phi}$, $\boldsymbol{\Xi}$, and $\boldsymbol{\Pi}$ depending on v and v' such that

$$\boldsymbol{X}(\boldsymbol{\epsilon}) = \boldsymbol{\Phi} \,.\, \boldsymbol{\epsilon} + \boldsymbol{\Xi} \,.\, (\mathbf{d}\mathsf{x} - \mathsf{v}\mathbf{d}t) + \boldsymbol{\Pi} \,.\, (\mathbf{d}\mathsf{v} - \mathsf{v}'\mathbf{d}t). \tag{9}$$

We also assert that A and k in (7) do not depend neither on t nor on x.

The identity (9) splits into more identities, obtained by evaluating the coefficients of the differentials $\mathbf{d}t$, $\mathbf{d}\mathsf{x}$, $\mathbf{d}\mathsf{v}$, and $\mathbf{d}\mathsf{v}'$ independently:

$$\begin{aligned} (\pi \,.\, \partial_{\mathsf{v}} + (\pi \cdot \mathsf{v})\, \mathsf{v} \,.\, \partial_{\mathsf{v}} + \Omega \cdot (\mathsf{v} \wedge \partial_{\mathsf{v}}))\, \mathsf{A} & \\ + 2\, (\pi \cdot \mathsf{v}) \mathsf{A} + (\mathsf{A}\mathsf{v}) \otimes \pi - \mathsf{A}\Omega = \boldsymbol{\Phi}\mathsf{A} \,; & \end{aligned} \tag{10}$$

$$2\, (\mathsf{A}\mathsf{v}') \otimes \pi + (\pi \cdot \mathsf{v}')\, \mathsf{A} = \boldsymbol{\Pi} \,; \tag{11}$$

$$-\mathsf{k} \otimes \pi = \boldsymbol{\Xi} \,; \tag{12}$$

$$X(\mathsf{k}) = \boldsymbol{\Phi}\mathsf{k} - \boldsymbol{\Xi}\mathsf{v} - \boldsymbol{\Pi}\mathsf{v}' \,. \tag{13}$$

In the above the '$\otimes$' symbol means the tensor (sometimes named as 'direct') product of matrices.

A skew-symmetric two-by-two matrix always has the inverse, so the 'Lagrange multipliers' $\mathbf{\Phi}$, $\mathbf{\Xi}$, and $\mathbf{\Pi}$ may explicitly be defined from the equations (10–12) and then substituted into (13). Subsequently, the equation (13) splits into the following identities by the powers of the variable v' and by the parameters $\mathbf{\Omega}$ and $\boldsymbol{\pi}$ (take notice of the derivative matrix $\mathsf{A}' = (\mathsf{v}' . \partial_{\mathsf{v}})\, \mathsf{A}$; also the vertical arrow sign points to the very last factor to which the aforegoing differential operator still applies):

$$(\Omega \cdot (\mathsf{v} \wedge \partial_{\mathsf{v}}))\, \mathsf{A}'\mathsf{v}' + (\Omega \cdot (\mathsf{v}' \wedge \partial_{\mathsf{v}}))\, \mathsf{A}\mathsf{v}' - (\mathsf{v}' . \partial_{\mathsf{v}})\, \mathsf{A}\Omega\mathsf{v}' \\ = (\Omega \cdot (\mathsf{v} \wedge \partial_{\mathsf{v}}))\, \overset{\downarrow}{\mathsf{A}}\mathsf{A}^{-1}\mathsf{A}'\mathsf{v}' - \mathsf{A}\Omega\mathsf{A}^{-1}\mathsf{A}'\mathsf{v}' \,; \tag{14}$$

$$(\Omega \cdot (\mathsf{v} \wedge \partial_{\mathsf{v}}))\, \mathsf{B} - \mathsf{B}\Omega = (\Omega \cdot (\mathsf{v} \wedge \partial_{\mathsf{v}}))\, \overset{\downarrow}{\mathsf{A}}\mathsf{A}^{-1}\mathsf{B} - \mathsf{A}\Omega\mathsf{A}^{-1}\mathsf{B} \,; \tag{15}$$

$$(\Omega \cdot (\mathsf{v} \wedge \partial_{\mathsf{v}}))\, \mathsf{c} = (\Omega \cdot (\mathsf{v} \wedge \partial_{\mathsf{v}}))\, \overset{\downarrow}{\mathsf{A}}\mathsf{A}^{-1}\mathsf{c} - \mathsf{A}\Omega\mathsf{A}^{-1}\mathsf{c} \,; \tag{16}$$

$$(\pi . \partial_{\mathsf{v}} + (\pi \cdot \mathsf{v})\, \mathsf{v} . \partial_{\mathsf{v}})\, \mathsf{A}'\mathsf{v}' + (\pi \cdot \mathsf{v})\, \mathsf{A}'\mathsf{v}' + (\pi \cdot \mathsf{v}')\, (\mathsf{v} . \partial_{\mathsf{v}})\, \mathsf{A}\mathsf{v}' + (\pi \cdot \mathsf{v}')\, \mathsf{A}'\mathsf{v} \\ = (\pi . \partial_{\mathsf{v}} + (\pi \cdot \mathsf{v})\, \mathsf{v} . \partial_{\mathsf{v}})\, \overset{\downarrow}{\mathsf{A}}\mathsf{A}^{-1}\mathsf{A}'\mathsf{v}' + (\pi\mathsf{A}^{-1}\mathsf{A}'\mathsf{v}')\, \mathsf{A}\mathsf{v} - 3\, (\pi \cdot \mathsf{v}')\, \mathsf{A}\mathsf{v}' \,; \tag{17}$$

$$(\pi . \partial_{\mathsf{v}} + (\pi \cdot \mathsf{v})\, \mathsf{v} . \partial_{\mathsf{v}})\, \mathsf{B} + (\mathsf{B}\mathsf{v}) \otimes \pi \\ = (\pi . \partial_{\mathsf{v}} + (\pi \cdot \mathsf{v})\, \mathsf{v} . \partial_{\mathsf{v}})\, \overset{\downarrow}{\mathsf{A}}\mathsf{A}^{-1}\mathsf{B} + (\mathsf{A}\mathsf{v}) \otimes \pi\mathsf{A}^{-1}\mathsf{B} + (\pi \cdot \mathsf{v})\mathsf{B} \,; \tag{18}$$

$$(\pi . \partial_{\mathsf{v}} + (\pi \cdot \mathsf{v})\, \mathsf{v} . \partial_{\mathsf{v}})\, \mathsf{c} \\ = (\pi . \partial_{\mathsf{v}} + (\pi \cdot \mathsf{v})\, \mathsf{v} . \partial_{\mathsf{v}})\, \overset{\downarrow}{\mathsf{A}}\mathsf{A}^{-1}\mathsf{c} + 3\, (\pi \cdot \mathsf{v})\, \mathsf{c} + (\pi\mathsf{A}^{-1}\mathsf{c})\, \mathsf{A}\mathsf{v} \,. \tag{19}$$

Cumbersome although routine calculations accompanying the simultaneous solving of the partial differential equations (14) and (17) with respect to the unknown function A_{12} produce the unique output of

$$\mathsf{A}_{12} = \frac{\mathrm{const}}{(1 + \mathsf{v}_1\mathsf{v}^1 + \mathsf{v}_2\mathsf{v}^2)^{3/2}} \,.$$

We remind that the system of the equations (14–19) and the system (6) must be solved simultaneously. Thus, the equation (6i) becomes trivial now.

Under the assumption of B being a symmetric matrix (see (6ii)), the solution of the equations $\{(15), (18)\}$ amounts to

$$\mathsf{B}_{ab} = \mathrm{const} \cdot (1 + \mathsf{v}{\cdot}\mathsf{v})^{-3/2} [\mathsf{v}_a\mathsf{v}_b - (1 + \mathsf{v}{\cdot}\mathsf{v})\, \mathsf{g}_{ab}] \,.$$

This automatically satisfies the equation (6iii) too. In what concerns the subsystem $\{(16), (19)\}$, only the trivial solution $\mathsf{c} = \mathbf{0}$ exists.

We are ready now to formulate the summary of the above development in terms of a proposition:

Proposition 1 *The invariant Euler-Poisson equation of a relativistic planar motion is:*

$$-\frac{*\mathsf{v}''}{(1 + \mathsf{v}{\cdot}\mathsf{v})^{3/2}} + 3\, \frac{*\mathsf{v}'}{(1 + \mathsf{v}{\cdot}\mathsf{v})^{5/2}}\, (\mathsf{v}{\cdot}\mathsf{v}') + \frac{\mu}{(1 + \mathsf{v}{\cdot}\mathsf{v})^{3/2}}\, [(1 + \mathsf{v}{\cdot}\mathsf{v})\, \mathsf{v}' - (\mathsf{v}'{\cdot}\mathsf{v})\, \mathsf{v}] = \mathbf{0} \,. \tag{20}$$

Arbitrary constant μ serves to parametrize the set of all the variational equations (20). The definition of the 'star operator' is common. Thus, $*1 = \mathbf{e}_{(1)} \wedge \mathbf{e}_{(2)}$ whereas $*(\mathbf{e}_{(1)} \wedge \mathbf{e}_{(2)}) = 1$ if the (pseudo)orthonormal frame $\{\mathbf{e}_{(1)}\,,\ \mathbf{e}_{(2)}\}$ carries the positive orientation; also $(*\mathbf{w})_a = \varepsilon_{ba}\mathrm{w}^b$. We recall for future use the definition of the inner product of two bi-vectors:

$$(\mathbf{a} \wedge \mathbf{b}) \cdot (\mathbf{c} \wedge \mathbf{d}) \;=\; (\mathbf{a} \cdot \mathbf{c}) \;-\; (\mathbf{b} \cdot \mathbf{d})\,.$$

Proposition 2 *The Euler-Poisson equation (20) describes the free motion of a spinning particle in two space dimensions if*

$$u_\nu S^{\mu\nu} = 0\,. \tag{21}$$

Demonstration. Equation (20) describes the world line of a particle parametrized by time. Passing to the proper time parametrization one obtains:

$$\ddot{\boldsymbol{u}} \times \boldsymbol{u} + \mu\dot{\boldsymbol{u}} = \mathbf{0}\,. \tag{22}$$

Let us introduce a vector $a_\mu = \frac{1}{2}\varepsilon_{\nu\lambda\mu}S^{\nu\lambda}$. Then the equation (2) takes the form

$$m_0\dot{\boldsymbol{u}} + \boldsymbol{a} \times \ddot{\boldsymbol{u}} = \mathbf{0} \tag{23}$$

with the consequence that

$$\boldsymbol{a} \cdot \dot{\boldsymbol{u}} = \mathbf{0}\,. \tag{24}$$

Any vector $\boldsymbol{a}$ may always be presented as

$$\boldsymbol{a} = (\boldsymbol{a} \cdot \boldsymbol{u})\, \boldsymbol{u} - (\boldsymbol{a} \times \boldsymbol{u}) \times \boldsymbol{u}\,. \tag{25}$$

If the solutions of the equation (22) are to satisfy also the equation (23), we may define the variable $\ddot{\boldsymbol{u}}$ from (22) with the help of $(\boldsymbol{u} \cdot \ddot{\boldsymbol{u}}) = -(\dot{\boldsymbol{u}} \cdot \dot{\boldsymbol{u}})$ and substitute it into (23) to obtain, in view of (24),

$$-\mu\, (\boldsymbol{a} \cdot \boldsymbol{u})\, \dot{\boldsymbol{u}} - m_0\dot{\boldsymbol{u}} + (\dot{\boldsymbol{u}} \cdot \dot{\boldsymbol{u}})\, \boldsymbol{a} \times \boldsymbol{u} = \mathbf{0}\,. \tag{26}$$

The condition (21) is equivalent to $\boldsymbol{a} \times \boldsymbol{u} = \mathbf{0}$, thus the equation (26) gives

$$(\boldsymbol{a} \cdot \boldsymbol{u}) = -m_0/\mu\,, \tag{27}$$

and from (25) it follows that

$$\boldsymbol{a} = -\frac{m_0}{\mu}\, \boldsymbol{u}\,, \tag{28}$$

so the equations (22) and (23) are now equivalent.

Remark 1. In most general setting, when $\|\mathfrak{u}\| \neq 1$ and $|\mathfrak{g}| \neq 1$ ($\mathfrak{g} = \det(\mathfrak{g}_{\mathfrak{mn}})$, $\mathfrak{m}$, $\mathfrak{n}$ run from 0 to 3) the four-vector of spin is introduced by

$$\mathfrak{s}_{\mathfrak{p}} = \frac{\sqrt{|\mathfrak{g}|}}{2\|\mathfrak{u}\|}\, \varepsilon_{\mathfrak{mnqp}}\mathfrak{u}^{\mathfrak{m}}\mathfrak{S}^{\mathfrak{nq}}\,. \tag{29}$$

It is straightforward that $\mathfrak{s}_3 = -a_\mu u^\mu$ and thus one gets a 'renormalization' of the spinning particle's mass:

$$\mu = \frac{m_0}{\mathfrak{s}_3}\,. \tag{30}$$

Remark 2. The first space-time curvature of the particle's world line governed by (22) is constant, i.e. $\dfrac{d}{d\tau}\|\dot{\boldsymbol{u}}\| = 0$.

Remark 3. The point symmetries of (20) are being exhausted by pseudo-orthogonal (resp. conformal) transformations if $m \neq 0$ (resp. $m = 0$). The proof may be found in [10].

We can present two different ($a = 1, 2$) Lagrange functions which produce the equation (20),

$$L_{(a)} = \frac{*(\mathbf{v}' \wedge \mathbf{e}_{(a)})}{(1 + \mathbf{v}{\cdot}\mathbf{v})^{1/2}(1 + \mathsf{g}_{aa}\|\mathbf{v} \wedge \mathbf{e}_{(a)}\|^2)}\mathsf{v}^a - \mu\,(1 + \mathbf{v}{\cdot}\mathbf{v})^{1/2}\,. \tag{31}$$

These differ by the total time derivative:

$$L_{(2)} - L_{(1)} = \frac{d}{dt}\arctan\frac{\mathsf{v}^1\mathsf{v}^2}{\sqrt{1 + \mathsf{v}_a\mathsf{v}^a}}\,.$$

Now it's time to pass over to the hamiltonian counterpart exposition.

3. HAMILTONIAN DESCRIPTION

A detailed exposition of the (generalized) hamiltonian theory applicable to the odd-order differential equations at no less extent than to the even-order ones may be found in [6]; a concise exposition is presented in [7].

A conjectural Legendre transformation is given by the momenta

$$\begin{aligned} \mathbf{p} &= \frac{\boldsymbol{\partial} L}{\boldsymbol{\partial}\mathbf{v}} - \frac{d}{dt}\frac{\boldsymbol{\partial} L}{\boldsymbol{\partial}\mathbf{v}'} \\ \mathbf{p}' &= \frac{\boldsymbol{\partial} L}{\boldsymbol{\partial}\mathbf{v}'}\,. \end{aligned}$$

The Hamilton function

$$H = -L + \mathbf{p}.\mathbf{v} + \mathbf{p}'.\mathbf{v}'$$

in case of a regular Legendre transformation gives rise to the following Hamilton system of first order equations:

$$-\frac{\partial H}{\partial \mathsf{x}^a} - \frac{\partial \mathsf{v}^b}{\partial \mathsf{x}^a}\frac{d}{dt}\mathsf{p}'_b - \frac{d}{dt}\mathsf{p}_a = 0\,; \tag{32}$$

$$-\frac{\partial H}{\partial \mathsf{p}_a} + \frac{d}{dt}\mathsf{x}^a = 0\,; \tag{33}$$

$$-\frac{\partial H}{\partial \mathsf{p}'_a} + \frac{\partial \mathsf{v}^a}{\partial \mathsf{x}^b}\frac{d}{dt}\mathsf{x}^b + \left(\frac{\partial \mathsf{v}^a}{\partial \mathsf{p}'_b} - \frac{\partial \mathsf{v}^b}{\partial \mathsf{p}'_a}\right)\frac{d}{dt}\mathsf{p}'_b = 0\,. \tag{34}$$

Both L_1 and L_2 from (31) produce one and the same Hamilton function:

$$\begin{aligned} H &= \frac{*(\mathbf{v} \wedge \mathbf{v}')}{(1+\mathbf{v}\cdot\mathbf{v})^{3/2}} + \mu(1+\mathbf{v}\cdot\mathbf{v})^{-1/2} \\ &= \mathbf{p}.\mathbf{v} + \mu\sqrt{1+\mathbf{v}\cdot\mathbf{v}}\,. \end{aligned} \tag{35}$$

This happens because both L_1 and L_2 lead to the same 'zero-order' momentum,

$$\mathbf{p} = \frac{*\mathbf{v}'}{(1+\mathbf{v}\cdot\mathbf{v})^{3/2}} - \mu\,\frac{\mathbf{v}}{\sqrt{1+\mathbf{v}\cdot\mathbf{v}}}\,. \tag{36}$$

However, the 'first-order' momentum $\mathbf{p}'$ has the identically equal to zero first (resp. second) component if one starts with L_1 (resp. L_2) alone. This suggests that we take

$$L = \frac{1}{2}(L_1 + L_2)$$

in an attempt to proceed further with a kind of regular Legendre transformation. In this case one calculates out the following expressions for the two components of the momentum $\mathbf{p}'$:

$$\mathsf{p}'_1 = \frac{\mathsf{v}^2}{2\sqrt{1+\mathbf{v}\cdot\mathbf{v}}\,(1+\mathsf{v}_1\mathsf{v}^1)}\,, \quad \mathsf{p}'_2 = -\,\frac{\mathsf{v}^1}{2\sqrt{1+\mathbf{v}\cdot\mathbf{v}}\,(1+\mathsf{v}_2\mathsf{v}^2)}\,. \tag{37}$$

In (34) the inverse to the Jacobi matrix of the Legendre transformation is indispensable to know (at least in its $\dfrac{\partial\mathbf{v}}{\partial\mathbf{p}'}$ part). The Jacobi matrix itself is readily obtained from (36) and (37), so we cite below only that part of its inverse, important for the forthcoming calculations,

$$\left(\frac{\partial \mathsf{v}^a}{\partial {\mathsf{p}'}_b}\right) = \frac{2}{\Delta}(1+\mathbf{v}\cdot\mathbf{v})^{3/2}\times$$

$$\times\begin{pmatrix} \dfrac{\mathsf{v}_2\mathsf{v}^1(3+3\,\mathsf{v}_2\mathsf{v}^2+2\,\mathsf{v}_1\mathsf{v}^1)}{(1+\mathsf{v}_2\mathsf{v}^2)^2} & -1 \\ 1 & -\dfrac{\mathsf{v}_1\mathsf{v}^2(3+3\,\mathsf{v}_1\mathsf{v}^1+2\,\mathsf{v}_2\mathsf{v}^2)}{(1+\mathsf{v}_1\mathsf{v}^1)^2} \end{pmatrix}, \tag{38}$$

where Δ denotes the determinant of the matrix in (38). The Reader may also easily convince himself by direct calculation that

$$\left(\frac{\partial \mathsf{v}^a}{\partial \mathsf{p}_{\ b}}\right) = \mathbf{0}\,. \tag{39}$$

As far as the Hamilton function (35) does not depend on the space and time variables, the essential part of the Hamilton system, the equation (32), constitutes nothing else but the conservation of the momentum,

$$\frac{d\mathbf{p}}{dt} = \mathbf{0}\,. \tag{40}$$

The remaining two equations mean merely that we pick up only the holonomic solutions of (40). Thus, the equation (33) in view of (39) reads

$$-\mathbf{v} + \frac{d\mathbf{x}}{dt} = \mathbf{0}\,,$$

and so it describes the kernel of the first one of the two differential forms in (8). The equation (34) in view of (38) reads

$$-\frac{\partial H}{\partial \mathbf{p}'} + 4\frac{(1+\mathbf{v}{\cdot}\mathbf{v})^{3/2}}{\Delta}\left(*\,\frac{d\mathbf{p}'}{dt}\right) = \mathbf{0}\,. \tag{41}$$

But we can calculate $\dfrac{\partial H}{\partial \mathbf{p}'}$ from (35) by means of (38) and also differentiate the definitions (37) explicitly. After substituting into (41) it becomes evident that the Hamilton equation (34) describes the kernel of the second differential form in (8). In fact the equation (41) transforms into the following system:

$$\mathbf{M}\,\boldsymbol{.}\,(d\mathbf{v} - \mathbf{v}'dt) = \mathbf{0}\,,$$

where $\mathbf{M}$ is the matrix from (38).

Let us return to the gravitational dipole particle. The Dixon system of equations in the general curved four-dimensional space-time accepts arbitrary parametrization of the particle's world line. It reads:

$$\frac{d\mathfrak{P}}{d\lambda} = \mathfrak{F} \tag{42}$$

$$\frac{d\mathfrak{S}}{d\lambda} = 2\,\mathfrak{P}\wedge\mathfrak{u}\,. \tag{43}$$

The force $\mathfrak{F}$ depends, apart from space and time variables, also upon the particle's tensorial spin $\mathfrak{S}$ and velocity $\mathfrak{u}$; it must satisfy the constraint $\mathfrak{F}\boldsymbol{.}\,\mathfrak{u} = \mathfrak{o}$. Once supplemented by the Pirani constraint (4), the system (42, 43) allows an equivalent transcription in terms of the spin four-vector (29), 'resolved' with respect to the variable $\mathfrak{P}$. Namely, in place of (43) we can write (renouncing the constraint $\|\mathfrak{u}\| = 1$)

$$\mathfrak{P} = \frac{m_0}{\|\mathfrak{u}\|}\,\mathfrak{u} - (\operatorname{sgn}\mathfrak{g})\;*\;(\dot{\mathfrak{u}}\wedge\mathfrak{u}\wedge\mathfrak{s})\,, \tag{44}$$

$$\dot{\mathfrak{s}}\wedge\mathfrak{u} = \mathfrak{o}\,.$$

In the parametrization by time ($\mathfrak{u}^0 = 1$), denoting by $\mathbf{P}$ the three-vector part of the four-vector $\mathfrak{P}$, the definition (44) takes up the shape

$$\mathbf{P} = m_0\,\frac{\mathbf{v}}{\sqrt{1+\mathbf{v}{\cdot}\mathbf{v}}} + \frac{\operatorname{sgn}\mathfrak{g}}{(1+\mathbf{v}{\cdot}\mathbf{v})^{3/2}}(\mathbf{v}'\times\mathbf{s} - \mathfrak{s}^0\,\mathbf{v}'\times\mathbf{v})\,, \tag{45}$$

where we use the notation $\mathbf{v}$ for the three-vector part of the velocity four-vector $\mathfrak{u}$ in the special case when $\mathfrak{u}^0 = 1$; also $\mathfrak{s} = (\mathfrak{s}^0, \mathbf{s})$ according to the general template.

Reassuming the particle's motion be two-dimensional only, one obtains from (45)

$$\mathbf{P} = \frac{m_0 \mathbf{v}}{\sqrt{1 + \mathbf{v}{\cdot}\mathbf{v}}} - (\operatorname{sgn} \mathfrak{g})\, \mathfrak{s}^3 \, \frac{*\mathbf{v}'}{(1 + \mathbf{v}{\cdot}\mathbf{v})^{3/2}}\,. \tag{46}$$

It suffices to compare (46) with the expression (36), assuming the notation (30), to come immediately to the conclusion that the quantity $\mathbf{P}$ coincides with $\mathfrak{s}^3$ times the canonical hamiltonian momentum.

Proposition 3 *The Hamilton function for the planar motion of free relativistic spinning particle with the mass (30) is given by (35).*

Remark 4. The Mathisson equation in general-relativistic framework and under arbitrary world line parametrization was considered in [11]. A set of the Lagrange functions to produce it in flat space-time was suggested in [12]. A second-order differential-geometric connection related to the equation (20) was constructed in [13].

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